

Cosmological Density Fluctuations in Stochastic Gravity – Formalism and Linear Analysis –

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We study primordial perturbations generated from quantum fluctuations of an inflaton based on the formalism of stochastic gravity. Integrating out the degree of freedom of the inflaton field, we analyze the time evolution of the correlation function of the curvature perturbation at tree level and compare it with the prediction made by the gauge-invariant linear perturbation theory. We find that our result coincides with that of the gauge-invariant perturbation theory if the e-folding from the horizon crossing time is smaller than some critical value ($\sim |\text{slow-roll parameter}|^{-1}$), which is the case for the scales of the observed cosmological structures. However, in the limit of the superhorizon scale, we find a discrepancy in the curvature perturbation, which suggests that we should include the longitudinal part of the gravitational field in the quantization of a scalar field even in stochastic gravity.

I. INTRODUCTION

Inflation has become the leading paradigm of the early universe not only because it solves the theoretical difficulties of the standard Big-Bang scenario but also because it explains the origin of the almost scale-invariant primordial density perturbations which have been found by the observation of the cosmic microwave background radiation (CMB). However, we may still not know some fundamental parts of the inflationary scenarios or models, mainly due to our ignorance of physics on the very short scale. It follows that we are largely interested in theoretical predictions about what we can learn about inflation from CMB [1, 2, 3, 4].

In order to analyze some inflation model by the observational data of CMB, it is necessary to evaluate the primordial perturbations generated during inflation. We believe that quantum fluctuation of an inflaton field gives the origin of seeds of cosmic large-scale structures. So it may be important to reveal how such a quantum fluctuation in the inflationary phase becomes classical density perturbations. So far there have been a few approaches [19, 20, 21, 22, 23, 24, 25, 26, 49, 50, 51, 52, 53]. Among them, stochastic gravity may provide one of the most systematic approaches [5, 6, 7, 8, 9, 10, 11]. It can describe a transition from quantum fluctuations to classical perturbations systematically. Stochastic gravity has been proposed in order to describe the behaviour of the gravitational field on the sub-Planck scale which is affected by quantum matter fields. On this energy scale, the quantum effect of the gravitational field may be ignored compared with quantum fluctuations of matter fields. Hence the gravitational field can be treated as a classical one. The semi-classical approach is justified.

In stochastic gravity, in order to find an effective action, we integrate out only the degree of matter fields by use of the

closed time path (CTP) [14, 15, 16, 17]. This is one way to perform coarse-graining [18]. As a result, we obtain the effective equation of motion for gravitational field under the influence of quantum fluctuation of matter fields, including those non-linear quantum effects. The obtained evolution equation is the Langevin type, which contains a stochastic source and a memory term.

Stochastic gravity looks similar to stochastic inflation, which was first proposed by Starobinsky [19, 20, 21, 22, 23, 24, 25, 26]. In stochastic inflation, in order to discuss the evolution of long wave modes of an inflaton scalar field, which play a crucial role in cosmological structure formation, the inflaton field is split into two parts: superhorizon modes and subhorizon ones. The long wave modes are affected by quantum fluctuations of the short wave modes. Then the effective equation of motion for the in-in expectation values of superhorizon modes can be derived by integrating out only subhorizon modes. In most works on stochastic inflation, the scalar field is discussed in a homogeneous and isotropic spacetime, and superhorizon modes and subhorizon ones interact with each other through a self-interaction of the scalar field.

In the last few years, some parts of quantum fluctuation of the gravitational field have been taken into account, by adopting a gauge-invariant variable as a canonical one to be quantized. This gauge-invariant variable includes not only a scalar field but also a longitudinal part of the gravitational field [27, 28, 29, 30]. Rigopoulos and Shellard have derived a non-linear evolution equation for long wave modes of the Sasaki-Mukhanov variable. The equation incorporates full non-linear dynamics on large scale. However, the contribution from quantum fluctuations of the short wave modes is evaluated based on the linear perturbation equations. Then only the effect from tree-level short wave modes is taken into account.

In stochastic gravity, on the other hand, although the way of the coarse-graining is different from that in stochastic inflation, the evolution equations in stochastic gravity are similar, and both equations can describe the transition from quantum fluctuations to classical perturbations. Furthermore, stochastic gravity can incorporate nonlinear quantum effects of a

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scalar field. This gives another advantage in stochastic gravity, as we will show below.

When we discuss only the leading part of the curvature perturbation (ζ) by means of linear analysis, in spite of mutual difference at the level of microphysics, it implies that most inflationary models may be compatible with the observational data. This is because this gauge invariant variable ζ , which is directly related to the temperature fluctuation of CMB, becomes constant in the superhorizon region [31, 32, 33]. In order to make a difference between many inflationary models, it is necessary to subtract more information from the observable. For this purpose, non-linear effects have been studied intensely [34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. Among such approaches to non-linear effects, stochastic gravity would be well-suited to compute the loop corrections induced from interaction between a scalar field and the gravitational field. It is because the CTP effective action includes also the non-linear effect of quantum fluctuations of a scalar field, and the effective equation makes it possible to discuss the loop corrections to primordial perturbations.

Roura and Verdaguer have applied the formalism of stochastic gravity to analyse the evolution of the primordial density perturbation [12, 13]. They discuss the evolution of Bardeen's gauge-invariant variable Φ under the approximation of de Sitter background spacetime. They showed that the order of the amplitude for Φ is the same as the prediction by quantization of the gauge-invariant variable. However, even at the tree level, so far, the time evolution of the curvature perturbation ζ in the superhorizon region, which is important to compare with the CMB observation, has not been sufficiently discussed in stochastic gravity. Hence, before we will discuss the non-linear effects, it may be better to reformulate how to evaluate the primordial perturbations in stochastic gravity. Since stochastic gravity includes additional effects such as a coarse-graining procedure to describe the transition from quantum fluctuations to classical perturbations, it is not trivial whether stochastic gravity predicts the same evolution of the primordial perturbations as that of the gauge-invariant approach. In this paper, we will consider the evolution of the curvature perturbation in a uniform density slicing. We find that stochastic gravity gives the same result as that in the gauge-invariant approach except for a limited case. The time evolution is characterized by the ratio of the Hubble horizon scale to the physical scale of fluctuation. We find that the amplitude of ζ in stochastic gravity deviates from the prediction of the gauge-invariant perturbation theory only when the ratio becomes nearly equal to zero. We discuss why two quantization procedures do not produce the same result. We will discuss loop corrections both to scalar and tensor perturbations in [44].

The paper is organized as follows. In Sec. II, we briefly summarize the quantization of gauge-invariant variables. In Sec. III, we present the basic idea of stochastic gravity, and consider the basic equations in stochastic gravity, i.e., the Einstein-Langevin equation, which describes the time evolution of the gravitational field affected by a quantum scalar field. Then we discuss the perturbations of the Einstein-Langevin equations around an inflationary background space-

time. In Sec. IV, we evaluate the correlation function of ζ and compare the result with the prediction in the gauge-invariant approach. Finally, in Sec. V, we discuss the reason why the prediction in stochastic gravity does not agree with that in the conventional linear theory in the superhorizon region.

Throughout in this paper, we consider a single-field inflation, which Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}[g^{ab}\partial_a\phi\partial_b\phi + 2V(\phi)], \quad (1.1)$$

where ϕ is an inflaton scalar field and $V(\phi)$ is its potential. To characterize the slow-roll inflation, we use two slow-roll parameters:

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \quad \text{and} \quad \eta_V \equiv \frac{V_{\phi\phi}}{\kappa^2 V}, \quad (1.2)$$

where $H = \dot{a}/a$ and $\kappa^2 \equiv 8\pi G$ are the Hubble expansion parameter and the reduced gravitational constant, respectively, and $V_{\phi\phi} \equiv d^2V/d\phi^2$. As time variable, we adopt the conformal time, $\tau (< 0)$, and represent the time derivative by a prime.

II. GAUGE-INVARIANT PERTURBATIONS

In this section, we briefly summarize a conventional approach for a quantization of an inflaton field, in which the gauge-invariant variable is used as a canonical variable. For more detailed explanations, please refer to the papers [35, 45, 46]. Throughout this paper, we follow the notation for perturbed variables defined in [47].

In a linear perturbation theory, each momentum mode decouples in the basic equations. Therefore, it is sufficient to consider only one mode with momentum k in the perturbed equations. As for the gauge condition, there are two convenient choices of time slicing to evaluate primordial perturbations. One is the slicing such that the spatial curvature perturbation vanishes, $\mathcal{R} = 0$, which is called the flat slicing. In this gauge choice, the equation of motion for fluctuation of a scalar field, is described as

$$\varphi_f'' + 2\mathcal{H}\varphi_f' + (k^2 + a^2V_{\phi\phi})\varphi_f + [2a^2V_{\phi}A - \phi'(A' + k\sigma_g)] = 0, \quad (2.1)$$

where φ_f , A and σ_g are perturbed variables of a scalar field, a lapse function and a shear of a unit normal vector field to a constant time hypersurface, respectively, and $\mathcal{H} \equiv a'/a = aH$ is a reduced Hubble parameter. Note that along with fluctuation of a scalar field, there appear metric perturbations such as A and σ_g [47]. In fact, the square bracket term in Eq. (2.1) represents fluctuation of the gravitational field. Using the perturbed Einstein equations

$$2[\mathcal{H}(A' + k\sigma_g) + 2(\mathcal{H}' + 2\mathcal{H}^2)A] = -2\kappa^2 a^2 V_{\phi}\varphi \quad (2.2)$$

$$\mathcal{H}(A' + k\sigma_g) = -\kappa^2 \varphi_f [a^2 V_{\phi} + \mathcal{H}\phi'(3 - \varepsilon)], \quad (2.3)$$

Eq. (2.1) can be rewritten by means of one variable, φ_f , as

$$\varphi_f'' + 2\mathcal{H}\varphi_f' + (k^2 + a^2 m_{\text{eff}}^2)\varphi_f = 0, \quad (2.4)$$

where

$$m_{\text{eff}}^2 \equiv V_{\phi\phi} + \kappa^2 \frac{\phi'}{\mathcal{H}} \left\{ 2V_\phi + \frac{\mathcal{H}\phi'}{a^2} (3 - \varepsilon) \right\}. \quad (2.5)$$

The contribution from fluctuation of the gravitational field is put together into the effective mass term. If we ignore the contribution from metric perturbations, together with $V_{\phi\phi}$, which are suppressed by the slow-roll parameter ε and η_V , φ_f obeys the evolution equation for a massless field in de Sitter spacetime, i.e.,

$$\varphi_f''(\tau) + 2\mathcal{H}\varphi_f'(\tau) + k^2\varphi_f(\tau) \simeq 0. \quad (2.6)$$

This is why this slicing is preferred in order to consider the evolution of perturbations in subhorizon region. In de Sitter space, we easily find a two-point correlation function for a massless scalar field. In order to determine positive frequency modes, we have to impose initial conditions. When we consider only linear perturbations, it is sufficient to discuss subhorizon modes on an initial time (τ_i), which are important for structure formation. In the subhorizon limit ($|k\tau| \gg 1$), it is appropriate to impose that this scalar field behaves as if it were a free field in Minkowski spacetime, whose mode functions are given as

$$\varphi_{f,\mathbf{k}}(\tau_i) = \frac{1}{\sqrt{2k}} e^{-ik\tau_i} \quad (2.7)$$

at the initial time of inflation. Then the two-point function for $|k\tau| \ll 1$ is obtained as

$$\begin{aligned} \langle \varphi_{f,\mathbf{k}}(\tau) \varphi_{f,\mathbf{p}}(\tau) \rangle &= (2\pi)^3 \delta(\mathbf{k} + \mathbf{p}) \frac{H^2}{2k^3} (1 + k^2 \tau^2) \\ &\simeq (2\pi)^3 \delta(\mathbf{k} + \mathbf{p}) \frac{H^2}{2k^3}. \end{aligned} \quad (2.8)$$

The other convenient gauge choice is the comoving slicing, on which the energy flux vanishes, $T_i^0 = 0$. It is preferred when we discuss the evolution of perturbations in the superhorizon region. Since the curvature perturbation in this slicing, \mathcal{R}_c , is related to φ_f as

$$\mathcal{R}_c = -\frac{H}{\dot{\phi}} \varphi_f, \quad (2.9)$$

the evolution equation for \mathcal{R}_c is obtained from Eq. (2.4) and Eq. (2.5) as

$$\mathcal{R}_{c,\mathbf{k}}''(\tau) + 2\frac{z'}{z}\mathcal{R}_{c,\mathbf{k}}'(\tau) + k^2\mathcal{R}_{c,\mathbf{k}}(\tau) = 0, \quad (2.10)$$

where

$$z \equiv \frac{a\dot{\phi}}{H} = \text{sgn}(\dot{\phi}) \frac{a\sqrt{2\varepsilon}}{\kappa}. \quad (2.11)$$

Note that this is the exact equation for \mathcal{R}_c . To derive this equation, neither the slow-roll approximation nor the long wave

approximation is imposed. When we consider the large-scale limit of $k \rightarrow 0$, \mathcal{R}_c includes a constant mode function. Using the approximation

$$\frac{z'}{z} \simeq -\frac{1 + 3\varepsilon - \eta_V}{\tau}, \quad (2.12)$$

we find the equation for \mathcal{R}_c as

$$\mathcal{R}_{c,\mathbf{k}}''(\tau) - \frac{2(1 + 3\varepsilon - \eta_V)}{\tau} \mathcal{R}_{c,\mathbf{k}}'(\tau) + k^2 \mathcal{R}_{c,\mathbf{k}}(\tau) = 0. \quad (2.13)$$

If we ignore the time evolution of the slow-roll parameters, this equation is solved by Hankel functions. We find general solution as

$$\mathcal{R}_c = x^\alpha \{c_1 H_\alpha^{(1)}(x) + c_2 H_\alpha^{(2)}(x)\}, \quad (2.14)$$

where $x \equiv -k\tau$ and $\alpha = 3/2 + 3\varepsilon - \eta_V$. x denotes the ratio of the horizon scale to the physical size of a perturbation with momentum k . Since both Hankel functions ($H_\alpha^{(1)}(x)$ and $H_\alpha^{(2)}(x)$) behave as $x^{-\alpha}$ when $x \ll 1$, we find that \mathcal{R}_c approaches to constant in the superhorizon region. Hence, it is sufficient to evaluate a two-point function of \mathcal{R}_c around the horizon-crossing time, i.e.,

$$\begin{aligned} \langle \mathcal{R}_c(\mathbf{k}, \eta) \mathcal{R}_c(\mathbf{k}', \eta') \rangle &= \left(\frac{H_k}{\dot{\phi}_k} \right)^2 \langle \varphi_f(\mathbf{k}, \eta) \varphi_f(\mathbf{k}', \eta') \rangle \\ &\simeq \frac{(2\pi)^3}{4k^3} \delta(\mathbf{k} + \mathbf{k}') \frac{(\kappa H_k)^2}{\varepsilon_k}, \end{aligned} \quad (2.15)$$

where the suffix k represents that the variables are evaluated at the time when the fluctuation mode with k crosses a horizon. It follows from Eq. (2.15) that the amplitude of the metric perturbation on given comoving scale, k , is determined by the energy density (or Hubble parameter H) and by deviation of the equation of state from de Sitter vacuum, which is described by the slow-roll parameter ε at the time of horizon crossing. Taking into account the fact that the Hubble parameter decreases and the slow-roll parameter increases as inflation goes on, the description Eq. (2.15) shows that inflation provides the red-tilted spectrum, which agrees with the CMB observation.

III. STOCHASTIC GRAVITY

In this section, first we shortly summarize the basic points on stochastic gravity. The basic equation in stochastic gravity describes the evolution of the gravitational field, whose source term is given by quantum matter fields. The effective action is obtained by integrating matter fields with the CTP formalism. We find the Langevin type equation, i.e., the so-called Einstein-Langevin equation which is analogous to the equation of motion for the Brownian particle. In Brownian motion, the deterministic trajectory is influenced and modified by stochastic behaviour of the environmental source. It is worth noting that this Langevin type equation is well-suited

not only to helping us understand the properties of inflation and the origin of large-scale structures in the Universe, but also to explaining the transition from quantum fluctuations to classical seeds.

The Langevin type equation is characterized by the existence of stochastic variables and memory terms. The stochastic variables represent quantum fluctuations of the matter fields. Under the Gaussian approximation, its statistics is described by the two-point function $N_{abc'd'}(x_1, x_2)$, which is called a Noise kernel. The Noise kernel is determined by the imaginary part of the CTP effective action. The CTP effective action with a coarse-graining includes such an imaginary part. The effective equation of motion is also derived from this effective action. It is usually a differential-integral equation representing the non-Markovian nature. The integral part, which is called the memory term, depends on the history of the gravitational field itself. In fact, its integrand is composed of fluctuations of the gravitational field δg_{ab} and the two-point function $H_{abc'd'}(x_1, x_2)$, which is called the Dissipation kernel. The Dissipation kernel is determined by the real part of the CTP effective action. In Sec. III B and Sec. III C, we will explicitly evaluate the Noise kernel and the Dissipation kernel. We also analyze the perturbations of the above Langevin type equation, i.e., the Einstein-Langevin equation. In Sec. IV, using the results obtained in this section, we will evaluate the correlation function of the primordial perturbations.

A. Einstein-Langevin equation

When we consider an interacting quantum system, which includes the gravitational field on the sub-Planck scale, we may expect that quantum fluctuation of the matter fields dominates that of the gravitational field. In stochastic gravity, we assume that the gravitational field is not quantized but classical because we are interested in a sub-Planckian scale. However, it is important to take into account fluctuation of a gravitational field which is induced through interaction with quantum matter fields. In order to discuss such dynamics of the gravitational field, the CTP formalism is helpful.

The effective action for the in-in expectation value of the gravitational field is derived by integrating matter fields. In [8], Martin and Verdaguer derived the effective equation of motion based on the CTP functional technique applied to a system-environment interaction, more specifically, on the influence functional formalism by Feynman and Vernon. This CTP effective action contains two specific terms, in addition to the ordinary Einstein-Hilbert action, describing the induced effects through interaction with a quantum scalar field. One is a memory term, by which the equation of motion depends on the history of the gravitational field itself. The other is a stochastic source ξ_{ab} , which describes quantum fluctuation of a scalar field. The latter is obtained from the imaginary part of the effective action, and as such it cannot be interpreted as a standard action. Indeed, there appear statistically weighted stochastic noises as a source for the gravitational field. Under the Gaussian approximation, this stochastic variable is characterized by the mean value and the two-point correlation func-

tion:

$$\begin{aligned}\langle \xi_{ab}(x) \rangle &= 0, \\ \langle \xi_{ab}(x_1) \xi_{c'd'}(x_2) \rangle &= N_{abc'd'}(x_1, x_2),\end{aligned}\quad (3.1)$$

where the bi-tensor $N_{abc'd'}(x_1, x_2)$ is the Noise kernel which represents quantum fluctuation of the energy-momentum tensor in a background spacetime, i.e.,

$$\begin{aligned}N_{abc'd'}(x_1, x_2) &\equiv \frac{1}{4} \text{Re}[F_{abc'd'}(x_1, x_2)] \\ &= \frac{1}{8} \langle \{ \hat{T}_{ab}(x_1) - \langle \hat{T}_{ab}(x_1) \rangle, \hat{T}_{ab}(x_2) - \langle \hat{T}_{ab}(x_2) \rangle \} \rangle [g],\end{aligned}\quad (3.2)$$

where $\{ \hat{X}, \hat{Y} \} = \hat{X}\hat{Y} + \hat{Y}\hat{X}$, g is the metric of a background spacetime, and the bi-tensor $F_{abc'd'}(x, y)$ is defined by

$$\begin{aligned}F_{abc'd'}(x_1, x_2) &\equiv \langle \hat{T}_{ab}(x_1) \hat{T}_{c'd'}(x_2) \rangle [g] \\ &\quad - \langle \hat{T}_{ab}(x_1) \rangle [g] \langle \hat{T}_{c'd'}(x_2) \rangle [g].\end{aligned}\quad (3.3)$$

The expectation value for the quantum scalar field is evaluated in the background spacetime g . The Noise kernel and the Dissipation kernel correspond to the contributions from internal lines or loops of the Feynman diagrams, which consist of propagators of the scalar field and do not include propagators of the gravitational field as internal lines. Including the above-mentioned stochastic source of ξ_{ab} , the effective equation of motion for the gravitational field is written as

$$G^{ab}[g + \delta g] = \kappa^2 \left[\langle \hat{T}^{ab} \rangle_R [g + \delta g] + 2\xi^{ab} \right], \quad (3.4)$$

where δg is the metric perturbation induced by quantum fluctuation of matter fields and stochastic source ξ_{ab} is characterized by the average value and the two-point correlation function Eq. (3.1).

Note that this equation is the same as the semiclassical Einstein equation except for a source term of stochastic variables ξ_{ab} , which represents quantum fluctuation of energy-momentum tensor of matter field. Furthermore, the expectation value of energy-momentum tensor includes a non-local effect as follows. It consists of three terms as

$$\begin{aligned}\langle \hat{T}^{ab} \rangle_R [g + \delta g] &= \langle \hat{T}^{ab}(x) \rangle [g] + \langle \hat{T}^{(1)ab}[\phi[g], \delta g](x) \rangle [g] \\ &\quad - 2 \int d^4y \sqrt{-g(y)} H^{abcd}[g](x, y) \delta g_{cd}(y) + O(\delta g^2),\end{aligned}\quad (3.5)$$

where $\hat{T}^{(1)ab}$ is the linearized energy momentum tensor and H^{abcd} is the Dissipation kernel. It depends both on the gravitational field and on a scalar field. The evolution equation for a scalar field also depends on the gravitational field. As a result, the expectation value of the energy-momentum tensor depends directly on the spacetime geometry and indirectly through a scalar field. When we perturb a spacetime with metric g to that with $(g + \delta g)$, two different changes appear in

the r.h.s. of Eq. (3.5). The second term represents the direct change, which is expressed in terms of fluctuation of the gravitational field δg as

$$\begin{aligned} \langle \hat{T}^{(1)ab}[\phi[g], \delta g](x) \rangle &= \left(\frac{1}{2} g^{ab} \delta g_{cd} - \delta_c^a g^{be} \delta g_{de} - \delta_c^b g^{ae} \delta g_{de} \right) \langle \hat{T}^{cd} \rangle[g] \\ &\quad - \left\{ \left(1 - \frac{2}{3} \varepsilon \right) \rho + \frac{\delta \psi^2}{2a^2} \right\} \left(g^{ac} g^{bd} - \frac{1}{2} g^{ab} g^{cd} \right) \delta g_{cd}. \end{aligned} \quad (3.6)$$

where $\delta \psi^2$ is defined by

$$\delta \psi^2 \equiv \langle \nabla_0 \psi \nabla_0 \psi + \gamma^{ij} \nabla_i \psi \nabla_j \psi \rangle[g] \quad (3.7)$$

We can neglect this term safely on the sub-Planck scale because this term is smaller by the order of $(\kappa H)^2$ than the previous term. To derive this expression, we have used the background evolution equation for a scalar field.

While the third integral term in the r.h.s. of Eq. (3.5) represents the effect from the indirect change and is characterized by the Dissipation kernel, which is given by

$$H_{abc'd'}(x_1, x_2) = H_{abc'd'}^{(S)}(x_1, x_2) + H_{abc'd'}^{(A)}(x_1, x_2) \quad (3.8)$$

$$H_{abc'd'}^{(S)}(x_1, x_2) = \frac{1}{4} \text{Im}[S_{abc'd'}(x_1, x_2)] \quad (3.9)$$

$$H_{abc'd'}^{(A)}(x_1, x_2) = \frac{1}{4} \text{Im}[F_{abc'd'}(x_1, x_2)], \quad (3.10)$$

where $S_{abc'd'}(x_1, x_2)$ is defined by

$$S_{abc'd'}(x_1, x_2) \equiv \langle T^* \hat{T}_{ab}(x_1) \hat{T}_{c'd'}(x_2) \rangle[g]. \quad (3.11)$$

T^* denotes that we take time ordering before we apply the derivative operators in the energy momentum tensor. As pointed out in [8], only if the background spacetime g satisfies the semiclassical Einstein equation, the gauge invariance of the Einstein-Langevin equation is guaranteed. Hence, in this paper, to guarantee the gauge invariance, we assume the background spacetime satisfies the semiclassical Einstein equation.

The Einstein-Langevin equation describes fluctuation of the gravitational field, which is induced by the quantum fluctuations of a scalar field. The Einstein-Langevin equation, given by Eq. (3.4) includes two different sources. One is a stochastic source ξ_{ab} , whose correlation function is given by the noise kernel. From the explicit form of a Noise kernel (3.2), we find that ξ_{ab} represents the quantum fluctuation of the energy momentum tensor. The other is an expectation value of the energy momentum tensor in the perturbed spacetime $(g + \delta g)$, which includes a memory term. The integrand of a memory term consists of a Dissipation kernel and fluctuation of the gravitational field. In order to investigate the evolution for fluctuation of the gravitational field, it is necessary to evaluate the quantum correction of a scalar field and calculate the Noise kernel and the Dissipation kernel.

B. Perturbed Einstein-Langevin equation

Since the CTP effective action contains the full quantum effect of matter fields, it is expected that we can also deal with non-linear quantum effects such as loop corrections by means of stochastic gravity. Before we consider such non-linear effects we shall discuss linear perturbations, which correspond to a tree-level effect in the Feynman diagram. Note that in a linear perturbation theory in the homogeneous and isotropic universe, three types of perturbations, scalar, vector, and tensor perturbations are decoupled from each other. Then, we discuss each perturbation independently.

We are interested in density perturbations as seeds for structure formation. Then, it is sufficient to consider only scalar perturbations. In this paper, we choose the metric as follows:

$$\begin{aligned} ds^2 &= -a^2(\tau)(1 + 2AY)d\tau^2 - 2a^2(\tau)\frac{k}{\mathcal{H}}\Phi Y_j d\tau dx^j \\ &\quad + a^2(\tau)\gamma_{ij}dx^i dx^j, \end{aligned} \quad (3.12)$$

where a and γ_{ij} are a scale factor and the 3 metric of maximally symmetric space, and \mathcal{A} and $(k/\mathcal{H})\Phi$ are the lapse function and the shift vector, respectively. The scalar perturbations can be expanded by a complete set of harmonic function $Y(\mathbf{x})$ on the three-dimensional space and Y_j is defined by $Y_j \equiv -k^{-1}Y_{|j}$. In this gauge choice, the spatial curvature perturbation vanishes. We will discuss the evolution of the following variable:

$$\zeta = \frac{1}{2\varepsilon} \delta_f, \quad (3.13)$$

where $\delta_f \equiv \delta\rho/\rho$ is density perturbation. This variable ζ is gauge-invariant, and turns out to be a curvature perturbation in a uniform density slicing. In the classical perturbation theory, the energy conservation law implies that this variable is conserved in the superhorizon region for a single-field inflation [31, 32, 33]. ζ is directly related to a gravitational potential at the late stage of the universe and in turn to the observed CMB fluctuations.

The density perturbation in the present slicing is given by

$$\begin{aligned} \delta T_0^0 &\equiv -\rho \delta_f Y \\ &= \delta g_{0c} \langle \hat{T}^{0c}(x) \rangle[g] + g_{0c} \{ \langle \hat{T}^{(1)0c}[\phi[g], \delta g](x) \rangle \\ &\quad - 2 \int d^4 y \sqrt{-g(y)} H^{0cde}[g](x, y) \delta g_{de}(y) + 2\xi^{0c} \}. \end{aligned} \quad (3.14)$$

Since the background energy-momentum tensor is given by

$$\langle \hat{T}^{0a}(x) \rangle[g] = g^{0b} \langle \hat{T}_b^a(x) \rangle[g] = -\rho g^{0a}, \quad (3.15)$$

the direct contribution from a change of the gravitational field is described as

$$\langle \hat{T}^{(1)00}[\phi[g], \delta g](x) \rangle = -\frac{2}{a^2} \left\{ 1 + \frac{\varepsilon}{3} + O((\kappa H)^2) \right\} \rho A Y. \quad (3.16)$$

With these two relations (3.14) and (3.16), the density perturbation in flat slicing is written as

$$\delta_f \simeq -\frac{2\varepsilon}{3}\mathcal{A} + \frac{2}{\rho}(\delta\rho_m + \delta\rho_\xi), \quad (3.17)$$

where we have defined the density perturbations of the stochastic source ξ_b^a and of the memory term as follows:

$$\begin{aligned} \delta\rho_m &\equiv \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \left[g^{00} \int d^4y \sqrt{-g(y)} \right. \\ &\quad \times H_{00c'd'}[g](x, y) g^{c'e'} g^{d'f'} \delta g_{e'f'}(y) \Big] \\ \delta\rho_\xi &\equiv \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \left[-g_{00} \xi^{00}(x) \right]. \end{aligned} \quad (3.18)$$

Since the contribution from the memory term, $\delta\rho_m$, contains the past history of metric perturbations, it seems very difficult to deal with such a term. However, as we will show in Sec. IV A, we can evaluate the memory term, rewriting the integral equation into a differential equation.

The Hamiltonian constraint equation gives a relation between the gauge-invariant variable \mathcal{A} and the density perturbation δ_f as

$$\mathcal{A} = \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi - \frac{\delta_f}{2}. \quad (3.19)$$

Using it, we eliminate \mathcal{A} in Eq. (3.17), and find

$$\left(1 - \frac{\varepsilon}{3} \right) \delta_f = \frac{2}{\rho} (\delta\rho_\xi + \delta\rho_m) + O\left((k/\mathcal{H})^2 \right). \quad (3.20)$$

Hence, in the superhorizon region, the two-point function for δ_f is expressed in terms of four correlation functions of $\delta\rho_\xi$ and $\delta\rho_m$, i.e.,

$$\begin{aligned} \langle \delta_f \mathbf{k}(\tau) \delta_f \mathbf{p}(\tau) \rangle &\simeq \frac{4}{V(\tau)^2} \left[\langle \delta\rho_\xi \mathbf{k}(\tau) \delta\rho_\xi \mathbf{p}(\tau) \rangle \right. \\ &\quad + \langle \delta\rho_m \mathbf{k}(\tau) \delta\rho_\xi \mathbf{p}(\tau) \rangle + \langle \delta\rho_\xi \mathbf{k}(\tau) \delta\rho_m \mathbf{p}(\tau) \rangle \\ &\quad \left. + \langle \delta\rho_m \mathbf{k}(\tau) \delta\rho_m \mathbf{p}(\tau) \rangle \right]. \end{aligned} \quad (3.21)$$

Here we have used the relation

$$V(\tau) = \left(1 - \frac{\varepsilon}{3} \right) \rho + O(\rho (\kappa H)^2). \quad (3.22)$$

The four correlation functions in Eq. (3.21) are described by the noise and the Dissipation kernels. We present the details in Appendix A. The correlation function for $\delta\rho_\xi$ is given from the Noise kernel as follows:

$$\begin{aligned} &\langle \delta\rho_\xi \mathbf{k}(\tau_1) \delta\rho_\xi \mathbf{p}(\tau_2) \rangle \\ &\simeq \frac{(2\pi)^3}{2} \kappa^2 \varepsilon V_1 V_2 \delta(\mathbf{k} + \mathbf{p}) \text{Re} \left[G_k^+(\tau_1, \tau_2) \right], \end{aligned} \quad (3.23)$$

where $G_k^+(\tau_1, \tau_2)$ is the Wightman function in momentum space, which will be given by the mode function. We have denoted $X(\tau_j)$ as X_j , where $X(\tau)$ is some function of a conformal time τ .

Since the Noise kernel is not a time-ordered correlation function, it should be described by the Wightman function in momentum space $G_k^+(\tau_1, \tau_2)$. In order to derive this relation, we have to assume that

$$\begin{aligned} &\left| \frac{\dot{\phi}}{a} \frac{\partial}{\partial \tau_1} \text{Re} \left[G_k^+(\tau_1, \tau_2) \right] \right|, \left| \frac{\dot{\phi}}{a} \frac{\partial}{\partial \tau_2} \text{Re} \left[G_k^+(\tau_1, \tau_2) \right] \right| \\ &<< \left| \alpha^{(1)} \kappa V \text{Re} \left[G_k^+(\tau_1, \tau_2) \right] \right|, \end{aligned} \quad (3.24)$$

where $\alpha^{(1)} \equiv V_\phi/(\kappa V)$. The Wightman function with the initial condition imposed in Sec. III C satisfies this relation in slow-roll inflationary universe.

While the density perturbation of the memory term is expressed as follows :

$$\begin{aligned} \delta\rho_m \mathbf{k}(\tau) &\simeq -2\varepsilon \kappa^2 V(\tau) \int_{\tau_k}^{\tau} d\tau_1 (a_1)^4 \\ &\times \left[\delta\rho_\xi \mathbf{k}(\tau_1) + \delta\rho_m \mathbf{k}(\tau_1) \right] \text{Im} \left[G_k^+(\tau, \tau_1) \right]. \end{aligned} \quad (3.25)$$

As mentioned in Appendix A, since contributions from sub-horizon region will oscillate, we can ignore them in the integral of Eq. (3.25). Hence the lower bound of the time integral in Eq. (3.25) may be given by the horizon crossing time, $\tau_k = -1/k$. Since the memory term depends on the past history, the integrand of the memory term contains $\delta\rho_m$ itself. In Sec. IV, rewriting this form to a differential equation, we evaluate contributions from the memory term.

Then, once the Wightman function, $G_k^+(\tau, \tau_2)$, is determined, we can evaluate the correlation function for δ_f and ζ from Eq. (3.21), (3.23), and (3.25). In the next subsection, imposing the appropriate initial condition, we will find the mode functions, and in turn the corresponding Green function.

C. Wightman function

Here we consider the Wightman function in momentum space,

$$G_k^+(\tau_1, \tau_2) \equiv \psi_{\mathbf{k}}(\tau_1) \psi_{\mathbf{k}}^*(\tau_2), \quad (3.26)$$

where $\psi_{\mathbf{k}}(\tau)$ is the mode function of a quantum scalar field in the inflationary universe. It satisfies the wave equation

$$\begin{aligned} &\psi_{\mathbf{k}}''(\tau) + 2\mathcal{H} \psi_{\mathbf{k}}'(\tau) \\ &+ \{k^2 + a^2 \kappa^2 V_V\} \psi_{\mathbf{k}}(\tau) = 0. \end{aligned} \quad (3.27)$$

We solve this equation under the slow-roll condition. Introducing a new variable as $\tilde{\psi}_{\mathbf{k}}(\tau) \equiv a(\tau) \psi_{\mathbf{k}}(\tau)$, this equation is rewritten as

$$\tilde{\psi}_{\mathbf{k}}''(\tau) + [k^2 - \{2 - \varepsilon - \eta_V(3 - \varepsilon)\} \mathcal{H}^2] \tilde{\psi}_{\mathbf{k}}(\tau) = 0, \quad (3.28)$$

where we have used the relation

$$a^2 \kappa^2 V = a^2 \kappa^2 \rho \left(1 - \frac{\varepsilon}{3} \right) = 3\mathcal{H}^2 \left(1 - \frac{\varepsilon}{3} \right). \quad (3.29)$$

In an inflationary stage on the sub-Planck scale, we can neglect the term whose magnitude is smaller by the order of

$(\kappa H)^2$ than that of the leading term. We also ignore non-linear terms with respect to the slow-roll parameters. So we do not take into account time evolution of the slow-roll parameters. Under these assumptions, the equation for $\tilde{\psi}$ becomes

$$\frac{d^2}{dx^2} \tilde{\psi}(x) + \left[1 - \frac{2 + 3(\varepsilon - \eta_V)}{x^2} \right] \tilde{\psi}(x) = 0, \quad (3.30)$$

where we have used $\mathcal{H} \simeq -1/[(1-\varepsilon)\tau]$. The general solution for this equation is given by the Hankel functions as

$$\tilde{\psi}_{\mathbf{k}}(\tau) = x^{\frac{1}{2}} \left[\tilde{C} H_{\beta}^{(1)}(x) + \tilde{D} H_{\beta}^{(2)}(x) \right], \quad (3.31)$$

where $\beta^2 \equiv 9/4 + 3(\varepsilon - \eta_V)$, with two arbitrary constants \tilde{C} and \tilde{D} . It implies

$$\psi_{\mathbf{k}}(\tau) = \frac{x^{\frac{1}{2}}}{a(\tau)} \left[\tilde{C} H_{\beta}^{(1)}(x) + \tilde{D} H_{\beta}^{(2)}(x) \right]. \quad (3.32)$$

We can assume that the mode functions should have the same form as in Minkowski spacetime, i.e.,

$$\psi_{\mathbf{k}}(\tau_i) = \frac{1}{\sqrt{2k}} e^{-ik\tau_i}, \quad (3.33)$$

when the wavelength is much shorter than the horizon scale, i.e., at very early times of the universe. This fact may be true in the present gauge rather than the comoving gauge. Then the mode function and the Wightman function in momentum space are given by

$$\psi_{\mathbf{k}}(\tau) = \frac{\sqrt{\pi|\tau|}}{2} \frac{a_i}{a(\tau)} e^{i\frac{(2\beta+1)\pi}{4}} H_{\beta}^{(1)}(x) \quad (3.34)$$

$$G_k^+(\tau_1, \tau_2) = \frac{\pi\sqrt{\tau_1\tau_2}}{4} \frac{a_i^2}{a_1 a_2} H_{\beta}^{(1)}(x_1) H_{\beta}^{(2)}(x_2), \quad (3.35)$$

where $x \equiv -k\tau$. Setting $a_i = 1$, the scale factor $a(\tau)$ is given by $a(\tau) = \left(\frac{\tau_i}{\tau}\right)^{1+\varepsilon}$. Using this expression, the Wightman function is rewritten as

$$\begin{aligned} G_k^+(\tau_1, \tau_2) &= \frac{\pi\sqrt{\tau_1\tau_2}}{4} \left(\frac{\tau_1\tau_2}{\tau_i^2}\right)^{1+\varepsilon} H_{\beta}^{(1)}(x_1) H_{\beta}^{(2)}(x_2) \\ &= \frac{\pi}{4} \frac{(x_1 x_2)^{\frac{3}{2}}}{k^3} \left(\frac{\tau_1\tau_2}{\tau_i^2}\right)^{\varepsilon} (1-\varepsilon)^2 H_i^2 H_{\beta}^{(1)}(x_1) H_{\beta}^{(2)}(x_2) \end{aligned} \quad (3.36)$$

Here we have used the relation

$$\tau_i^{-2} = (1-\varepsilon)^2 \mathcal{H}_i^2 = (1-\varepsilon)^2 H_i^2. \quad (3.37)$$

In order to compute the correlation functions for $\delta\rho_{\xi}$ and $\delta\rho_m$, it is sufficient to consider the evolution of the Wightman function in the superhorizon region. The behaviour of $G_k^+(\tau_1, \tau_2)$ in the superhorizon region is given in Appendix B. Substituting the approximated expression (B5) for the real part of the

Wightman function into Eq. (3.23), we find that the correlation for the density perturbation of stochastic variable $\delta\rho_{\xi}$ is expressed as

$$\begin{aligned} \langle \delta\rho_{\xi\mathbf{k}}(\tau_1) \delta\rho_{\xi\mathbf{p}}(\tau_2) \rangle \\ \simeq \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) \varepsilon V_1 V_2 \frac{\kappa^2 H_i^2}{k^3} \frac{(x_1 x_2)^{\eta_V}}{x_i^{2\varepsilon}}. \end{aligned} \quad (3.38)$$

Similarly, substituting the approximated expression (B6) for the imaginary part of the Wightman function into Eq. (3.25), we find that the density perturbation for the memory term is

$$\begin{aligned} \delta\rho_{m\mathbf{k}}(\tau) &\simeq \frac{\varepsilon V \kappa^2}{3} \frac{x^{\frac{1}{2}}}{ak^2} \int_x^1 dx_1 (a_1)^3 x_1^{\frac{1}{2}} \\ &\times \left\{ \left(\frac{x_1}{x}\right)^{\beta} - \left(\frac{x}{x_1}\right)^{\beta} \right\} \{ \delta\rho_{\xi\mathbf{k}}(\tau_1) + \delta\rho_{m\mathbf{k}}(\tau_1) \} \\ &\simeq \varepsilon H^2 \frac{x^{\frac{3}{2}+\varepsilon} x_i^{2(1+\varepsilon)}}{k^2} \int_x^1 dx_1 x_1^{-\frac{5}{2}-3\varepsilon} \\ &\times \left\{ \left(\frac{x_1}{x}\right)^{\beta} - \left(\frac{x}{x_1}\right)^{\beta} \right\} \{ \delta\rho_{\xi\mathbf{k}}(\tau_1) + \delta\rho_{m\mathbf{k}}(\tau_1) \}. \end{aligned} \quad (3.39)$$

IV. CORRELATION FUNCTIONS IN STOCHASTIC GRAVITY

As shown in Eq. (3.21), the correlation function for the density perturbation in flat slicing consists of four correlation functions :

$$f_{\xi\xi}(\tau_1, \tau_2) \equiv \langle \delta\rho_{\xi\mathbf{k}}(\tau_1) \delta\rho_{\xi\mathbf{p}}(\tau_2) \rangle \quad (4.1)$$

$$f_{\xi m}(\tau_1, \tau_2) \equiv \langle \delta\rho_{\xi\mathbf{k}}(\tau_1) \delta\rho_{m\mathbf{p}}(\tau_2) \rangle \quad (4.2)$$

$$f_{m\xi}(\tau_1, \tau_2) \equiv \langle \delta\rho_{m\mathbf{k}}(\tau_1) \delta\rho_{\xi\mathbf{p}}(\tau_2) \rangle \quad (4.3)$$

$$f_{mm}(\tau_1, \tau_2) \equiv \langle \delta\rho_{m\mathbf{k}}(\tau_1) \delta\rho_{m\mathbf{p}}(\tau_2) \rangle. \quad (4.4)$$

Here we have not explicitly shown the momentum dependence in those expressions just for simplicity. $f_{\xi\xi}$ has already been given in Eq. (3.38), i.e.,

$$f_{\xi\xi}(\tau_1, \tau_2) \simeq \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) \varepsilon V_1 V_2 \frac{\kappa^2 H_i^2}{k^3} \frac{(x_1 x_2)^{\eta_V}}{x_i^{2\varepsilon}}. \quad (4.5)$$

The other three functions contain the contribution from the memory term. In the next subsection, IV A, we present these three correlation functions. Then, these functions determine the correlation functions for δ_f (or ζ), which also gives the correlation function for the curvature perturbation in uniform density slicing. This gauge-invariant variable ζ is related to the fluctuation of the temperature of CMB.

A. Memory term

The memory term represents the non-Markovian nature of the effective equation of motion. The evolution equation depends on the past history, showing its non-local nature. The

memory term describes interesting phenomena, such as dissipation. It plays an important role in non-equilibrium systems. Especially, in stochastic gravity, the memory term represents the indirect dependence on the gravitational field in the energy momentum tensor. This point has already been emphasised in Sec. III A. The analysis, however, bothers us due to the complexity of the integral equation. Here, rewriting this integral equation into the differential equation, we evaluate the contribution from the memory term.

The time evolution of the density perturbation for the memory term $\delta\rho_m$ is described by Eq. (3.39). It determines the correlation function for $\delta\rho_m$ in the integral form. The correlation function between $\delta\rho_\xi$ and $\delta\rho_m$ satisfies the integral equation:

$$f_{m\xi}(x, x_2) = \varepsilon H^2 \frac{x^{\frac{3}{2}+\varepsilon} x_i^{2(1+\varepsilon)}}{k^2} \int_x^1 dx_1 x_1^{-\frac{5}{2}-3\varepsilon} \times \left[\left(\frac{x_1}{x} \right)^\beta - \left(\frac{x}{x_1} \right)^\beta \right] \cdot [f_{\xi\xi}(x_1, x_2) + f_{m\xi}(x_1, x_2)]. \quad (4.6)$$

Taking derivatives of this equation twice, we find a differential equation as

$$\frac{d}{dx} \left\{ x^{-2-2(\varepsilon+\eta_V)} \frac{d}{dx} \left[x^{-2\varepsilon-\eta_V} f_{m\xi}(x, x_2) \right] \right\} = 2\beta\varepsilon x^{-4-4\varepsilon+\eta_V} [f_{\xi\xi}(x, x_2) + f_{m\xi}(x, x_2)]. \quad (4.7)$$

We can simplify it to the second order differential equation with a source term, i.e.,

$$\frac{d^2}{dx^2} f_{m\xi}(x, x_2) - \frac{2}{x} \frac{d}{dx} f_{m\xi}(x, x_2) + \frac{3(\varepsilon + \eta_V)}{x^2} f_{m\xi}(x, x_2) = \frac{3\varepsilon}{x^2} f_{\xi\xi}(x, x_2). \quad (4.8)$$

Since Eq. (4.6) is the integral equation, we have to impose the initial conditions, i.e.,

$$f_{m\xi}(1, x_2) = \frac{d}{dx} f_{m\xi}(x, x_2) \Big|_{x=1} = 0. \quad (4.9)$$

The time $x = 1$ corresponds to the horizon crossing time. Using two independent solutions for the homogeneous equations, which are the power-law functions

$$f_1(x) \equiv x^{3-(\varepsilon+\eta_V)}, \quad f_2(x) \equiv x^{\varepsilon+\eta_V}, \quad (4.10)$$

we find the solution for Eq. (4.8) with initial conditions (4.9) as

$$f_{m\xi}(x, x_2) = \varepsilon \int_x^1 \frac{dx_1}{x_1} \times \left[\left(\frac{x}{x_1} \right)^{\varepsilon+\eta_V} - \left(\frac{x_1}{x} \right)^{3-(\varepsilon+\eta_V)} \right] f_{\xi\xi}(x_1, x_2). \quad (4.11)$$

Substituting $f_{\xi\xi}$ given in Eq. (4.5) into Eq. (4.11), we obtain the correlation function for $\delta\rho_m$ and $\delta\rho_\xi$ as

$$f_{m\xi}(\tau_1, \tau_2) = \langle \delta\rho_m \mathbf{k}(\tau_1) \delta\rho_\xi \mathbf{p}(\tau_2) \rangle^{(2)} \simeq \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) \varepsilon V_1 V_2 \frac{\kappa^2 H_k^2}{k^3} (x_1 x_2)^{\eta_V} \times \left[(x_1^{-\varepsilon} - 1) + \frac{\varepsilon}{3} \{ (x_1)^3 - 1 \} \right], \quad (4.12)$$

where we have used the fact that the Hubble parameter at the initial time τ_i is related to that at the horizon crossing time as $H_i^2 x_i^{-2\varepsilon} \simeq H_k^2$. Note that $f_{\xi m}(\tau_1, \tau_2) = f_{m\xi}(\tau_2, \tau_1)$.

As for the correlation function for $\delta\rho_m$, we have the integral equation,

$$f_{mm}(x, x_2) = \varepsilon H^2 \frac{x^{\frac{3}{2}+\varepsilon} x_i^{2(1+\varepsilon)}}{k^2} \int_x^1 dx_1 x_1^{-\frac{5}{2}-3\varepsilon} \times \left[\left(\frac{x_1}{x} \right)^\beta - \left(\frac{x}{x_1} \right)^\beta \right] \cdot [f_{\xi m}(x_1, x_2) + f_{mm}(x_1, x_2)]. \quad (4.13)$$

It is converted into the equivalent differential equation,

$$\frac{d^2}{dx^2} f_{mm}(x, x_2) - \frac{2}{x} \frac{d}{dx} f_{mm}(x, x_2) + \frac{3(\varepsilon + \eta_V)}{x^2} f_{mm}(x, x_2) \simeq \frac{3\varepsilon}{x^2} f_{\xi m}(x, x_2) \quad (4.14)$$

with the initial conditions,

$$f_{mm}(1, x_2) = \frac{d}{dx} f_{mm}(x, x_2) \Big|_{x=1} = 0. \quad (4.15)$$

Using the time evolution of $f_{\xi m}(x_1, x_2)$, we determine the time evolution of $f_{mm}(x_1, x_2)$ as follows:

$$f_{mm}(x_1, x_2) = \varepsilon \int_{x_1}^1 \frac{dx}{x} \left[\left(\frac{x_1}{x} \right)^{\varepsilon+\eta_V} - \left(\frac{x}{x_1} \right)^{3-(\varepsilon+\eta_V)} \right] \times f_{\xi m}(x, x_2) \simeq \frac{(2\pi)^3}{4} \delta^{(3)}(\mathbf{k} + \mathbf{p}) \varepsilon V_1 V_2 \frac{\kappa^2 H_k^2}{k^3} (x_1 x_2)^{\eta_V} \times \left[(x_1^{-\varepsilon} - 1) + \frac{\varepsilon}{3} \{ (x_1)^3 - 1 \} \right] \times \left[(x_2^{-\varepsilon} - 1) + \frac{\varepsilon}{3} \{ (x_2)^3 - 1 \} \right]. \quad (4.16)$$

Here, we find the following specific property of the memory term. The leading term of the equal time correlation function, $f_{mm}(\tau, \tau)$, is given by

$$f_{mm}(\tau, \tau) \simeq \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) \varepsilon V^2 \frac{\kappa^2 H_k^2}{k^3} \times x^{2\eta_V} \left[(x^{-\varepsilon} - 1) - \frac{\varepsilon}{3} \right]^2. \quad (4.17)$$

After this fluctuation crosses the horizon scale, until $-\varepsilon \log x \approx 1$, $x^{-\varepsilon}$ is nearly equal to unity. Hence the contributions from the memory term are suppressed by the slow-roll parameter ε , compared to $f_{\xi\xi}(\tau, \tau)$. In other words, as long as the e-foldings from the horizon crossing time to time τ , which is given by $N_{k \rightarrow \tau} \simeq -\log x$, is smaller than $1/\varepsilon$, the contribution from the memory term is negligible in the correlation function for δ_f (or ζ).

However, once the e-foldings $N_{k \rightarrow \tau}$ gets larger than $1/\varepsilon$, the term $x^{-\varepsilon}$ becomes much larger, and eventually the contribution from the memory term dominates the direct contribution from the stochastic variable $f_{\xi\xi}(\tau, \tau)$. In this case, the correlation function is approximated as

$$f_{mm}(\tau, \tau) \simeq \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) \varepsilon V^2 \frac{\kappa^2 H_k^2}{k^3} x^{2(\eta_V - \varepsilon)}. \quad (4.18)$$

B. Correlation functions

Substituting the correlation function for $\delta\rho_\zeta$ and $\delta\rho_m$, given by Eq. (4.5), Eq. (4.12) and Eq. (4.16) into Eq. (4.5), we determine the correlation function for the density perturbation in flat-slicing, δ_f , as

$$\langle \delta_f \mathbf{k}(\tau) \delta_f \mathbf{p}(\tau) \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{p}) \varepsilon \frac{\kappa^2 H_k^2}{k^3} x^{2\eta_V} \times \left[x^{-\varepsilon} + \frac{\varepsilon}{3}(x^3 - 1) \right]^2. \quad (4.19)$$

The density perturbation in flat-slicing, δ_f , is related to the curvature perturbation in uniform density slicing, ζ , by Eq. (3.13). We also find the correlation function of ζ as

$$\begin{aligned} \langle \zeta_{\mathbf{k}}(\tau) \zeta_{\mathbf{p}}(\tau) \rangle &\simeq \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) \frac{\kappa^2 H_k^2}{\varepsilon k^3} \\ &\quad \times x^{2\eta_V} \left[x^{-\varepsilon} + \frac{\varepsilon}{3}(x^3 - 1) \right]^2 \\ &\simeq \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) \frac{\kappa^2 H_k^2}{\varepsilon k^3} x^{2(\eta_V - \varepsilon)}. \end{aligned} \quad (4.20)$$

Note that after the horizon crossing time, until the e-folding $N_{k \rightarrow \tau}$ approaches to $1/|\eta_V - \varepsilon|$, the correlation function for ζ is approximated as

$$\langle \zeta_{\mathbf{k}}(\tau) \zeta_{\mathbf{p}}(\tau) \rangle \simeq \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) \frac{\kappa^2 H_k^2}{\varepsilon k^3}. \quad (4.21)$$

Then, in this region, the amplitude of the curvature perturbation ζ is constant. However, once the e-folding $N_{k \rightarrow \tau}$ exceeds $1/|\eta_V - \varepsilon|$, it will deviate from the constant value.

In order to compare our result with the prediction from quantization of the gauge-invariant variable summarized in Sec. II, it is helpful to show the relation

$$v - B = -\frac{2}{9(1+w)} \left(\frac{k}{\mathcal{H}} \right)^2 \left(\sigma_g - \frac{k}{\mathcal{H}} \zeta \right), \quad (4.22)$$

which is derived from the momentum constraint in the uniform density slicing. This relation implies that in the superhorizon region, the uniform density slicing agrees with the comoving slicing[48]. Hence, the correlation function for the curvature perturbation ζ in the gauge invariant perturbation theory is given by that for \mathcal{R}_c , i.e., Eq. (2.15). Comparing Eq. (4.20) to Eq. (2.15), we find that as long as the e-folding from the horizon crossing time is smaller than $1/|\eta_V - \varepsilon|$, our result is the same result as that in the gauge-invariant perturbation theory. However, when the e-folding $N_{k \rightarrow \tau}$ becomes larger than $1/|\eta_V - \varepsilon|$, the predictions by these two approaches are inconsistent.

In the next section, we discuss the reason why this deviation appears, and why the amplitude in stochastic gravity does not predict a constant evolution if the curvature perturbation is in the superhorizon region.

V. SUMMARY AND DISCUSSIONS

In this paper, we have considered the evolution of the primordial perturbations in a slow-roll inflationary universe. Primordial density perturbations, which play a crucial role at the later structure formation stage, are generated inside the horizon from the quantum fluctuation of a scalar field in the beginning of inflation, and then stretched out to the superhorizon scale.

However, since the observable quantity through the CMB observation is classical, in order to constrain an inflation model by the observation, it is necessary to consider a transition from quantum fluctuations to classical perturbations. Coarse-graining is the well-suited way to understand such a transition. Hence we have discussed cosmological perturbations in the inflationary stage in stochastic gravity. Based on the naive expectation that quantum fluctuation of the gravitational field is sufficiently small at the sub-Planck scale, we assume in stochastic gravity that contributions from the Feynman diagrams in which the gravitational field propagates as an internal line are negligibly small. Then, when we compute the CTP effective action, we integrate out only the dynamical degree of freedom of a scalar field. As a result, from this coarse-grained effective action, the evolution equation for the gravitational field is derived, which is called the Einstein-Langevin equation.

Although this effective equation makes it possible to discuss non-linear quantum effects such as the loop corrections of a scalar field, in this paper, we have first considered the tree-level effect, which is the leading contributor to the primordial perturbations. We have derived two-point correlation functions for density perturbations and the curvature perturbations. We find that our results are consistent with those in the gauge-invariant perturbation theory unless the e-folding from the horizon crossing time exceeds some critical value ($1/|\eta_V - \varepsilon|$).

However, as pointed out in the previous section, the curvature perturbation ζ does not keep constant in the superhorizon region if the e-folding gets beyond the critical value, in contrast to the prediction of the gauge-invariant perturbation theory. We expect that such a deviation appears due to neglect of quantum fluctuation of the gravitational field. In stochastic gravity, we do not integrate out the degree of freedom of quantum fluctuation of the gravitational field in the path integral of the CTP effective action. In this sense, quantum fluctuation of the gravitational field is neglected, although we include the fluctuation of the gravitational field induced by the quantum fluctuations of matter fields. It seems that this missing part induces the deviation in the superhorizon region.

In order to understand this point more clearly, it is helpful to reconsider the time evolution of the mode functions, $\psi_{\mathbf{k}}$, which equation is given by Eq. (3.27). We emphasize that this equation represents the wave equation for quantum scalar field in the background inflationary spacetime.

Comparing Eq. (3.27) with Eq. (2.1) for fluctuation of a scalar field in flat slicing (φ_f), we find that the equation for φ_f contains the fluctuation of the gravitational field. Such a contribution plays a crucial role in keeping the curvature perturbation

tion constant in the superhorizon region. In a gauge-invariant theory, the curvature perturbation in comoving slicing, \mathcal{R}_c is related to φ_f as

$$\mathcal{R}_c = -\frac{H}{\dot{\phi}}\varphi_f, \quad (5.1)$$

and the equation for \mathcal{R}_c is given by Eq. (2.11).

In comoving slicing, \mathcal{R}_c is the only one dynamical degree of freedom, and we have a single equation for it without neglect of any perturbed variables. The equation guarantees that the curvature perturbation \mathcal{R}_c is constant at a superhorizon scale. It means that only when all contributions of perturbations including fluctuation of the gravitational field are taken into account, $\zeta=\text{const}$ in the superhorizon region is guaranteed. If any part of the contributions is ignored, it implies the deviation from the constant evolution of ζ in the superhorizon region.

In fact, when we transform ψ into $\psi_c = (H/\dot{\phi})\psi$, imitating the relation between $\delta\phi_f$ and \mathcal{R}_c , the equation for ψ_c is written as

$$\begin{aligned} \psi_{c,\mathbf{k}}''(\tau) + 2\frac{z'}{z}\psi_{c,\mathbf{k}}'(\tau) \\ + \left[k^2 - 2\mathcal{H}^2\varepsilon \left(3 - \varepsilon - \frac{V_\phi}{\dot{\phi}H} \right) \right] \psi_{c,\mathbf{k}}(\tau) = 0 \end{aligned} \quad (5.2)$$

Unlike the case of \mathcal{R}_c , because of the existence of the term proportional to ε in the coefficient of $\psi_{c,\mathbf{k}}$, $\psi_{c,\mathbf{k}}=\text{constant}$ is no longer the solution for the superhorizon scale ($x \ll 1$), although the deviation is very small in a slow-roll inflation.

Finally, we consider the reason why neglect of quantum fluctuation of the gravitational field has influenced the behaviour of perturbations in the superhorizon region. In this discussion, first we have to distinguish fluctuation of the longitudinal mode of the gravitational field induced by matter fields from fluctuation of gravitons.

The latter may be very small and can be ignored at least on the sub-Planckian energy scale, while the former may not be so, because it is difficult to distinguish fluctuation of a scalar field from that of the longitudinal part of the gravitational field. In fact, if we change the gauge condition, they are mixed up. Hence when we quantize a scalar field, it may be natural to include the longitudinal gravitational field as well

[54]. In the present calculation based on stochastic gravity, we ignore both quantum fluctuations of the gravitational field. For the sub-Planck scale inflation, however, even in stochastic gravity, we should include contributions from fluctuation of the longitudinal mode which couples to the matter field, as we discussed above. We may have to improve our formulation for density perturbations in stochastic gravity. However, we should emphasize that the deviation in the present approach from the gauge-invariant approach becomes large only when the e-foldings from the horizon crossing time to the definite time exceed $1/|\varepsilon - \eta_V|$. For example, if we assume that $|\varepsilon - \eta_V|$ is around 0.01, then it implies that this deviation appears only when the e-foldings from the horizon crossing time to the end of inflation $N_{k \rightarrow e}$ exceeds about one hundred. Taking into account that e-folding $N_{k \rightarrow e}$ for today's Hubble horizon scale is about fifty, we may conclude that neglect of the longitudinal modes will influence only on a larger scale than the observed region today.

In stochastic gravity, as we mentioned, we can calculate the loop corrections, because the CTP effective action includes also the non-linear effect of quantum fluctuations of a scalar field, and the effective equation makes it possible to discuss the loop corrections to primordial perturbations. We will discuss it in [44].

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APPENDIX A: NOISE KERNEL AND DISSIPATION KERNEL

In Appendix A, we present the Noise kernel and the Dissipation kernel, which are given by the bi-tensors, $F_{abc'd'}(x_1, x_2)$ and $S_{abc'd'}(x_1, x_2)$. These expressions are given from integration of the CTP effective action in terms of fluctuation of a scalar field.

As seen in [44], the lowest-order part of these bi-tensors can be given by

$$\begin{aligned}
F_{abc'd'}(x_1, x_2) = & a_1 a_2 \dot{\phi}_1 \dot{\phi}_2 \{ \delta_a^0 \delta_{c'}^0 G_{;bd'}^+ + \delta_a^0 \delta_{d'}^0 G_{;bc'}^+ + \delta_b^0 \delta_{c'}^0 G_{;ad'}^+ + \delta_b^0 \delta_{d'}^0 G_{;ac'}^+ \\
& + \eta_{c'd'} (\delta_a^0 G_{;b0'}^+ + \delta_b^0 G_{;a0'}^+) + \eta_{ab} (\delta_{c'}^0 G_{;0d'}^+ + \delta_{d'}^0 G_{;0c'}^+) + \eta_{ab} \eta_{c'd'} G_{;00'}^+ \} \\
& - \eta_{ab} (a_1)^2 \kappa V_1 \alpha_1^{(1)} a_2 \dot{\phi}_2 (\delta_{c'}^0 G_{;d'}^+ + \delta_{d'}^0 G_{;c'}^+ + \eta_{c'd'} G_{;0}^+) \\
& - \eta_{c'd'} (a_2)^2 \kappa V_2 \alpha_2^{(1)} a_1 \dot{\phi}_1 (\delta_a^0 G_{;b}^+ + \delta_b^0 G_{;a}^+ + \eta_{ab} G_{;0}^+) \\
& + \eta_{ab}^x \eta_{c'd'}^y (a_1 a_2)^2 \kappa^2 V_1 V_2 \alpha_1^{(1)} \alpha_2^{(1)} G^+
\end{aligned} \tag{A1}$$

$$\begin{aligned}
S_{abc'd'}(x_1, x_2) = & a_1 a_2 \dot{\phi}_1 \dot{\phi}_2 \{ \delta_a^0 \delta_{c'}^0 (iG_F)_{;bd'} + \delta_a^0 \delta_{d'}^0 (iG_F)_{;bc'} + \delta_b^0 \delta_{c'}^0 (iG_F)_{;ad'} + \delta_b^0 \delta_{d'}^0 (iG_F)_{;ac'} \\
& + \eta_{c'd'} (\delta_a^0 (iG_F)_{;b0'} + \delta_b^0 (iG_F)_{;a0'}) + \eta_{ab} (\delta_{c'}^0 (iG_F)_{;0d'} + \delta_{d'}^0 (iG_F)_{;0c'}) + \eta_{ab} \eta_{c'd'} (iG_F)_{;00'} \} \\
& - \eta_{ab} (a_1)^2 \kappa V_1 \alpha_1^{(1)} a_2 \dot{\phi}_2 (\delta_{c'}^0 (iG_F)_{;d'} + \delta_{d'}^0 (iG_F)_{;c'} + \eta_{c'd'} (iG_F)_{;0}) \\
& - \eta_{c'd'} (a_2)^2 \kappa V_2 \alpha_2^{(1)} a_1 \dot{\phi}_1 (\delta_a^0 (iG_F)_{;b} + \delta_b^0 (iG_F)_{;a} + \eta_{ab} (iG_F)_{;0}) \\
& + \eta_{ab}^x \eta_{c'd'}^y (a_1 a_2)^2 \kappa^2 V_1 V_2 \alpha_1^{(1)} \alpha_2^{(1)} (iG_F),
\end{aligned} \tag{A2}$$

where $\alpha^{(1)} \equiv V_\phi / \kappa V \simeq \text{sgn}(V_\phi) \sqrt{2\varepsilon}$. There appear the two different Green functions such as $G^+ = G^+(x_1, x_2)$ and $G_F = G_F(x_1, x_2)$. The former is the Wightman function, and the latter is the Feynman function. To compute the correlation function $\delta\rho_\xi$ and $\delta\rho_m$, we have to know $F_{00'0'}(x_1, x_2)$, $F_{00'0'}(x_1, x_2)$, $S_{00'0'}(x_1, x_2)$, and $S_{00'0'}(x_1, x_2)$, which are written by

$$F_{00'0'}^0(x_1, x_2) = -(a_2)^2 O_{\tau_1} O_{\tau_2} G^+(x_1, x_2) \tag{A3}$$

$$F_{00'l'}^0(x_1, x_2) = -a_2 \dot{\phi}_2 O_{\tau_1} G^+(x_1, x_2)_{;l'} \tag{A4}$$

$$S_{00'0'}^0(x_1, x_2) = -(a_2)^2 \left[\theta(\tau_1 - \tau_2) O_{\tau_1} O_{\tau_2} G^+(x_1, x_2) + \theta(\tau_2 - \tau_1) O_{\tau_1} O_{\tau_2} G^+(x_2, x_1) \right] \tag{A5}$$

$$S_{00'l'}^0(x_1, x_2) = -a_2 \dot{\phi}_2 \left[\theta(\tau_1 - \tau_2) O_{\tau_1} G^+(x_1, x_2)_{;l'} + \theta(\tau_2 - \tau_1) O_{\tau_1} G^+(x_2, x_1)_{;l'} \right], \tag{A6}$$

where we have defined the operator O_τ as

$$O_\tau \equiv \frac{\dot{\phi}}{a(\tau)} \frac{\partial}{\partial \tau} + \alpha^{(1)} \kappa V. \tag{A7}$$

Using Eq. (A3)-(A6), we shall evaluate the noise and Dissipation kernels in order.

1. Noise kernel

Substituting the expression Eq. (A3) into Eq. (3.2), we obtain

$$\langle \xi_0^0(x_1) \xi_{0'}^{0'}(x_2) \rangle = N_0^{0'0'}(x_1, x_2) = \frac{1}{4} \text{Re}[F_{0'0'}^0(x_1, x_2)] = \frac{1}{4} O_{\tau_1} O_{\tau_2} \text{Re}[G^+(x_1, x_2)]. \tag{A8}$$

Hence we find the correlation function for $\delta\rho_\xi$ as

$$\begin{aligned}
\langle \delta\rho_{\xi\mathbf{k}}(\tau_1) \delta\rho_{\xi\mathbf{p}}(\tau_2) \rangle &= \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2 e^{-i\mathbf{k}\cdot\mathbf{x}_1} e^{-i\mathbf{p}\cdot\mathbf{x}_2} \langle \xi_0^0(x_1) \xi_{0'}^{0'}(x_2) \rangle^{(2)} \\
&= \frac{(2\pi)^3}{4} \delta(\mathbf{k} + \mathbf{p}) O_{\tau_1} O_{\tau_2} \text{Re} \left[G_k^+(\tau_1, \tau_2) \right] \\
&\simeq \frac{(2\pi)^3}{2} \delta(\mathbf{k} + \mathbf{p}) \varepsilon \kappa^2 V_1 V_2 \text{Re} \left[G_k^+(\tau_1, \tau_2) \right].
\end{aligned} \tag{A9}$$

In the last equality, we have approximated the operator as $O_\tau \simeq \alpha^{(1)} \kappa V$, because the time derivative of the Wightman function $\partial_\tau G^+$ is proportional to the slow-roll parameter η_V (see Appendix B).

2. Dissipation kernel

Substituting the expression Eq. (A3) and Eq. (A5) into Eq. (3.10) and Eq. (3.9), we obtain

$$\begin{aligned}
 H^{(2)0}_{00'0'}(x_1, x_2) &= H_A^{(2)0}_{00'0'}(x_1, x_2) + H_S^{(2)0}_{00'0'}(x_1, x_2) \\
 &= \frac{1}{4} \text{Im} \left[F^{(2)0}_{00'0'}(x_1, x_2) + S^{(2)0}_{00'0'}(x_1, x_2) \right] \\
 &= -\frac{(a_2)^2}{2} \theta(\tau_1 - \tau_2) O_{\tau_1} O_{\tau_2} \text{Im} \left[G^+(x_1, x_2) \right].
 \end{aligned} \tag{A10}$$

Similarly, substituting the expression Eq. (A4) and Eq. (A6) into Eq. (3.10) and Eq. (3.9), we find

$$\begin{aligned}
 H^{(2)0}_{00'l'}(x_1, x_2) &= H_A^{(2)0}_{00'l'}(x_1, x_2) + H_S^{(2)0}_{00'l'}(x_1, x_2) \\
 &= \frac{1}{4} \text{Im} \left[F^{(2)0}_{00'l'}(x_1, x_2) + S^{(2)0}_{00'l'}(x_1, x_2) \right] \\
 &= -\frac{a_2}{2} \dot{\phi}_2 \theta(\tau_1 - \tau_2) O_{\tau_1} \text{Im} \left[G^+(x_1, x_2)_{;l'} \right].
 \end{aligned} \tag{A11}$$

In order to evaluate $\delta\rho_{m,\mathbf{k}}(\tau)$, it is convenient to introduce the Dissipation kernel in momentum space as

$$\begin{aligned}
 H_{abc'd'}(\tau_1, \tau_2, \mathbf{p}, \mathbf{q}) &= \delta^{(3)}(\mathbf{p} + \mathbf{q}) \tilde{H}_{abc'd'}(\tau_1, \tau_2, |\mathbf{p}|) \\
 &= \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2 e^{-i\mathbf{p}\cdot\mathbf{x}_1} e^{-i\mathbf{q}\cdot\mathbf{x}_2} H_{abc'd'}(x_1, x_2).
 \end{aligned} \tag{A12}$$

Especially, the components, $(0, 0, 0', 0')$ and $(0, 0, 0', l')$, are given by

$$\begin{aligned}
 \tilde{H}^{(2)0}_{00'0'}(\tau_1, \tau_2, k) &= -\frac{(a_2)^2}{2} (2\pi)^3 \theta(\tau_1 - \tau_2) O_{\tau_1} O_{\tau_2} \text{Im} \left[G_k^+(\tau_1, \tau_2) \right] \\
 \tilde{H}^{(2)0}_{00'l'}(\tau_1, \tau_2, k) &= -\frac{a_2}{2} \dot{\phi}_2 (2\pi)^3 \theta(\tau_1 - \tau_2) \frac{k_{l'}}{i} O_{\tau_1} \text{Im} \left[G_k^+(\tau_1, \tau_2) \right].
 \end{aligned} \tag{A13}$$

Using this expression, $\delta\rho_{m,\mathbf{k}}(\tau)$ is written as

$$\begin{aligned}
 \delta\rho_{m\mathbf{k}}^{(2)}(\tau) &\simeq -2 \int \frac{d\tau_1}{(2\pi)^3} (a_1)^2 \left[\tilde{H}^{(2)0}_{00'0'}(\tau, \tau_1, k^a) \mathcal{A}_{\mathbf{k}}(\tau_1) + i \frac{k_{l'}}{k} \tilde{H}^{(2)0}_{00'l'}(\tau, \tau_1, k^a) \frac{k}{\mathcal{H}} \Phi_{\mathbf{k}}(\tau_1) \right] \\
 &= \int_{\tau_0}^{\tau} d\tau_1 (a_1)^4 \left[\frac{1}{3} \left(\frac{k}{\mathcal{H}_1} \right)^2 \left\{ O_{\tau} + 3 \frac{\dot{\phi}}{a} \mathcal{H}_1 \right\} \Phi_{\mathbf{k}}(\tau_1) - \frac{1}{2} O_{\tau} \delta_{f\mathbf{k}}(\tau_1) \right] O_{\tau_1} \text{Im} \left[G_k^+(\tau, \tau_1) \right],
 \end{aligned} \tag{A14}$$

where we have neglected the contribution from tensor perturbations which is expected to be much smaller than that from scalar perturbations ($\delta_{f\mathbf{k}}$ and $\Phi_{\mathbf{k}}$), and for the second equality, we have used the relation (3.19).

We can neglect the first term in Eq. (A14) as follows. In superhorizon region ($x_1 < 1$), this term is suppressed by the factor of $(k/\mathcal{H}_1)^2$. Around horizon crossing time ($x_1 \simeq 1$), this term is suppressed by the slow-roll parameter. Neglecting this term and using the approximation of $O_{\tau} \simeq \alpha^{(1)} \kappa V$, we find the density perturbation from the memory term $\delta\rho_m$ as

$$\begin{aligned}
 \delta\rho_{m\mathbf{k}}^{(2)}(\tau) &\simeq -\frac{1}{2} \int_{\tau_k}^{\tau} d\tau_1 (a_1)^4 \delta_{f\mathbf{k}}(\tau_1) O_{\tau} O_{\tau_1} \text{Im} \left[G_k^+(\tau, \tau_1) \right] \\
 &\simeq -\frac{1}{2} \int_{\tau_k}^{\tau} d\tau_1 (a_1)^4 \delta_{f\mathbf{k}}(\tau_1) 2\varepsilon \kappa^2 V V_1 \text{Im} \left[G_k^+(\tau, \tau_1) \right] \\
 &\simeq -2\varepsilon \kappa^2 V \int_{\tau_k}^{\tau} d\tau_1 (a_1)^4 \{ \delta\rho_{\xi\mathbf{k}}(\tau_1) + \delta\rho_{m\mathbf{k}}(\tau_1) \} \text{Im} \left[G_k^+(\tau, \tau_1) \right].
 \end{aligned} \tag{A15}$$

The lower bound of this integration is given by the horizon crossing time $\tau_k = -1/k$ because in the subhorizon region ($x_1 > 1$), the Hankel function appearing in $G_k^+(\tau, \tau_1)$ oscillates, and then no accumulative contribution exists.

APPENDIX B: WIGHTMAN FUNCTION IN THE SUPERHORIZON REGION

In this section, we will consider the evolution of the Wightman function in the superhorizon region, which is given by

$$G_k^+(\tau_1, \tau_2) = \frac{\pi}{4} \frac{\sqrt{\tau_1 \tau_2}}{a_1 a_2} H_\beta^{(1)}(x_1) H_\beta^{(2)}(x_2). \quad (\text{B1})$$

Since the Hankel function is expressed in terms of the Bessel function as follows :

$$H_\beta^{(1)}(x) = \left\{ H_\beta^{(2)}(x) \right\}^* = \frac{-i}{\sin \beta \pi} [J_{-\beta}(x) - e^{-\beta \pi i} J_\beta(x)]. \quad (\text{B2})$$

The Bessel function is expanded by a power series of x in the region of $0 < x < 1$, i.e.,

$$J_\beta(x) = \left(\frac{x}{2}\right)^\beta \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(\beta + m + 1)}. \quad (\text{B3})$$

Then we obtain the asymptotic behaviour of the Hankel function in the superhorizon region as

$$\begin{aligned} \text{Re}[H_\beta^{(1)}(x_1) H_\beta^{(2)}(x_2)] &\simeq \frac{1}{[\sin \beta \pi \Gamma(1 - \beta)]^2} \left(\frac{x_1 x_2}{4}\right)^{-\beta} \\ \text{Im}[H_\beta^{(1)}(x_1) H_\beta^{(2)}(x_2)] &\simeq \frac{\cos(\beta - 3/2)\pi}{\sin^2 \beta \pi} \frac{1}{\Gamma(1 - \beta) \Gamma(1 + \beta)} \left[\left(\frac{x_2}{x_1}\right)^\beta - \left(\frac{x_1}{x_2}\right)^\beta \right]. \end{aligned} \quad (\text{B4})$$

Using these formulas, the asymptotic behaviour of the Wightman function in the superhorizon region is given by

$$\text{Re}[G_k^+(\tau_1, \tau_2)] \simeq \frac{M_r}{(1 - \varepsilon)^2} \frac{\sqrt{\tau_1 \tau_2}}{a_1 a_2} (x_1 x_2)^{-\beta} \simeq M_r \frac{H_i^2}{k^3} (x_1 x_2)^{\frac{3}{2}} \left(\frac{\tau_1 \tau_2}{\tau_i^2}\right)^\varepsilon (x_1 x_2)^{-\beta} \quad (\text{B5})$$

$$\text{Im}[G_k^+(\tau_1, \tau_2)] \simeq M_i \frac{\sqrt{\tau_1 \tau_2}}{a_1 a_2} \left[\left(\frac{x_2}{x_1}\right)^\beta - \left(\frac{x_1}{x_2}\right)^\beta \right], \quad (\text{B6})$$

where the coefficients M_r and M_i are defined by

$$M_r \equiv \frac{\pi}{4} (1 - \varepsilon)^2 \frac{2^{2\beta}}{[\sin \beta \pi \Gamma(1 - \beta)]^2} = \frac{1}{2} + O(\varepsilon, \eta_V) \quad (\text{B7})$$

$$M_i \equiv \frac{\pi}{4} \frac{\cos(\beta - 3/2)\pi}{\sin^2 \beta \pi} \frac{1}{\Gamma(1 - \beta) \Gamma(1 + \beta)} = -\frac{1}{6} + O(\varepsilon, \eta_V). \quad (\text{B8})$$

In order to evaluate the correlation function of $\delta\rho_\xi$, it is also necessary to compute the time derivatives of the real part of the Wightman function, which are given by

$$\partial_{\tau_1} \text{Re}[G_k^+(\tau_1, \tau_2)] \simeq \frac{\eta_V}{\tau_1} G_k^+(\tau_1, \tau_2), \quad (\text{B9})$$

$$\partial_{\tau_2} \text{Re}[G_k^+(\tau_1, \tau_2)] \simeq \frac{\eta_V}{\tau_2} G_k^+(\tau_1, \tau_2), \quad (\text{B10})$$

$$\partial_{\tau_1} \partial_{\tau_2} \text{Re}[G_k^+(\tau_1, \tau_2)] \simeq \frac{\eta_V^2}{\tau_1 \tau_2} G_k^+(\tau_1, \tau_2). \quad (\text{B11})$$

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